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von Kármán–Howarth equation for magnetohydrodynamics and its consequences on third-order longitudinal structure and correlation functions

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A derivation in variable dimension of the scaling laws for mixed third-order longitudinal structure and correlation functions for incompressible magnetized flows is given for arbitrary correlation between the velocity and magnetic field with full isotropy, homogeneity, and incompressibility assumed. When close to equipartition between kinetic and magnetic energy, the scaling relations involve only structure functions in a manner similar to the "$\frac{4}{5}$ law" of Kolmogorov. [S1063-651X(98)50601-0]

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Magnetized flows are common in the universe, and are often in a turbulent state; examples are that of the solar corona, where magnetohydrodynamics (MHD) turbulence is likely at the origin of the power-law distributions of the luminosity of flares [1], or the interstellar medium where turbulence plays an essential role to sustain dense cold clouds [2]; in the solar wind, several in situ measurements (e.g., from the spacecrafts Helios, Voyager, and now Ulysses) provide information on structure functions of the velocity and magnetic fields indicating that the fields are strongly intermittent [3]; finally, the ground-based instrument Themis, soon operational, will provide similar information for the small magnetic structures of the solar photosphere. In view of these various domains of application, a theoretical basis equivalent to the rigorous results of von Kármán and Howarth [4], and of Kolmogorov [5] for incompressible nonhelical flows—concerning the temporal evolution of the longitudinal second-order energy tensor in terms of the third-order one, and the scaling of the latter in the inertial range—but for magnetized flows is in demand. The technically helpful restriction to incompressible nonhelical flows is unrealistic (although often used) as far as the above-mentioned astrophysical and geophysical flows are concerned [6]. Nevertheless, this work represents a first step in providing theoretical constraints on the dynamical evolution of small-scale structures in MHD. These structures (magnetic flux tubes and vortex and current sheets) have been studied analytically (e.g., in the context of reconnection), mostly in the linear regime, and have also been investigated numerically mostly in the incompressible case (e.g., [7] in two space dimensions, and in three dimensions [8]).

In 1941, Kolmogorov wrote three oft-quoted papers [5,9] concerning homogeneous isotropic incompressible fluid turbulence viz. (a) the distribution of kinetic energy among Fourier modes $E^V(k)\sim(\overline{\epsilon})^{2/3}k^{-5/3}$, where $\overline{\epsilon}$ is the kinetic energy flux to small scale (and decay rate $-\dot{E}^V$); (b) the temporal decay of the kinetic energy $E^V(t)\sim(t-t_*)^{-10/7}$ (where t_* is typically the time at which the enstrophy $\langle\boldsymbol{\omega}^2\rangle$ reaches its first maximum, with $\boldsymbol{\omega}=\boldsymbol{\nabla}\times\mathbf{v}$ the vorticity and $\mathbf{v}$ the velocity); and (c) the scaling law for the longitudinal third-order structure function $S_3(\mathbf{r})=-\frac{4}{5}\overline{\epsilon r}$, where $S_3(r)=\langle\delta v_L^3(\mathbf{r})\rangle$ with $\delta v_L(\mathbf{r})\equiv[\mathbf{v}(\mathbf{x}+\mathbf{r})-\mathbf{v}(\mathbf{x})]\cdot\mathbf{r}$, $\mathbf{r}$ being the displacement vector and $r=|\mathbf{r}|$; hereafter, the subscript L denotes longitudinal components. Whereas the first two laws stem from phenomenological considerations, the third one is deduced rigorously from the Navier-Stokes equations, in the inertial range neglecting viscosity. Extensions of the so-called "$\frac{4}{5}$" law of Kolmogorov to the case of a scalar passively advected [10], such as the temperature or a pollutant in the atmosphere, as well as to the case of a scalar with a dynamical effect on the velocity, such as the magnetic potential in two-dimensional MHD [11], show that it is the correlation between the scalar and the velocity field that is constrained by the invariance properties of the equations, thereby indicating the necessity not to neglect such correlations in modeling turbulent flows, since they are essential for nonzero turbulent transfer.

We write for an incompressible conducting flow

$$(\partial_t+\mathbf{z}^{\mp}\cdot\boldsymbol{\nabla})\mathbf{z}^{\pm}=-\boldsymbol{\nabla}P_*+\nu_+\nabla^2\mathbf{z}^{\pm}+\nu_-\nabla^2\mathbf{z}^{\mp} \qquad (1)$$

for the Elsässer fields $\mathbf{z}^{\pm}=\mathbf{v}\pm\mathbf{b}$, where $\mathbf{b}$ is the magnetic induction, $P_*=P+b^2/2$ the total pressure, $\nu_{\pm}=(\nu\pm\eta)/2$, where ν is the viscosity and η the magnetic diffusivity, and $\boldsymbol{\nabla}\cdot\mathbf{v}=0$, $\boldsymbol{\nabla}\cdot\mathbf{b}=0$; a force term can be added as well.

In conducting flows, the nonlinear interactions and thus the ensuing phenomenology describing such interactions are more complex than for neutral fluids. The standard model of Iroshnikov and Kraichnan [12] leads to $E^T(k)\sim(\epsilon^T B_0)^{1/2}k^{-3/2}$ for the energy Fourier spectrum and, in three dimensions in the simplest case, a decay of energy that is substantially slower than for neutral fluids with $E^T(t)\sim(t-t_*)^{-5/6}$ [13], where B_0 is a large-scale magnetic field, E^T is the total (kinetic plus magnetic) energy, and $\epsilon^T=-\dot{E}^T$ is its flux. The phenomenological modeling of Ref. [12] relies on taking into account the slowing down of the energy transfer due to the interaction of $\mathbf{z}^{\pm}$ eddies propagating in opposite directions along B_0.

The Kolmogorov law [5] arises from the conservation of energy in the limit of negligible dissipation, i.e., in the inertial range (and then taking the limit $r\to 0$, otherwise $S_3(\mathbf{r})\sim r^3$ trivially from a Taylor expansion); the $\frac{4}{5}$ law follows from the von Kármán–Howarth equation, which we first generalize to MHD using the fact that the invariants in terms of the Elsässer fields are $E^{\pm}=\langle|\mathbf{z}^{\pm}|^2\rangle/2$. In the derivation, a few steps of a kinematical nature concerning the properties of tensors are needed; the algebra is somewhat lengthy but the principles are well known.

One proceeds in two steps [14]. We first seek to derive the equation for the temporal evolution of the second-order correlation tensors, which in the full isotropic case can be written as [14]

$$R_{ij}^{\pm}(\mathbf{r}) \equiv \langle z_i^{\pm}(\mathbf{x}) z_j^{\pm}(\mathbf{x}')\rangle = F^{\pm}(r) r_i r_j + G^{\pm}(r)\delta_{ij},$$

where $\mathbf{x}' \equiv \mathbf{x}+\mathbf{r}$. Since the MHD equations are symmetric in the exchange of the $\pm$ variables, it suffices to work with, say, $\mathbf{z}^+$. Particularizing the correlation tensor now to the + case, its longitudinal $[f^+(r)]$ and lateral $[g^+(r)]$ coefficients can be written as usual as $z^{+^2} f^+(r) = \langle z_L^+(\mathbf{x}) z_L^+(\mathbf{x}')\rangle$, and $z^{+^2} g^+(r) = \langle z_n^+(\mathbf{x}) z_n^+(\mathbf{x}')\rangle$; isotropy implies equipartition of the energy of the longitudinal (L) and lateral (n) components of $\mathbf{z}^+$, namely $\langle z_L^{+^2}\rangle = \langle z_n^{+^2}\rangle = d_*^{-1}\langle z_i^+ z_i^+\rangle \equiv z^{+^2}$, where $d_* = d$ is the number of components of the vectors that are retained in the dynamical evolution of the flow [15]; a similar definition holds for $(z^-)^2$. In space dimension d, incompressibility yields $g^+(r) = f^+(r) + r f'^+(r)/(d-1)$.

Proceeding in a similar manner for the two-point third-order correlation tensor between the $\mathbf{z}^+$ and $\mathbf{z}^-$ fields, we consider its longitudinal component

$$\mathcal{C}_{LLL}^{+-+}(\mathbf{r}) \equiv \langle z_L^+(\mathbf{x}) z_L^-(\mathbf{x}) z_L^+(\mathbf{x}')\rangle \equiv C_3^+ k^{+-+}(r),$$

where $C_3^+ = z^{+^2} z^-$. Using incompressibility and homogeneity again, the von Kármán–Howarth equation for MHD is now derived after a few lines of algebra as

$$\partial_t [z^{+^2} f^+(r)] = \left(\frac{\partial}{\partial r} + \frac{d+1}{r}\right) C_3^+ k^{+-+}(r) + 2\left(\frac{\partial^2}{\partial r^2} + \frac{d+1}{r}\frac{\partial}{\partial r}\right)[\nu_+ z^{+^2} f^+(r) + \nu_- z^+ z^- f^{+-}(r)],$$

where $z^+ z^- f^{+-}(r) \equiv \langle z_L^+(\mathbf{x}) z_L^-(\mathbf{x}')\rangle = z^+ z^- f^{-+}(r)$ is the longitudinal coefficient of the cross correlator between the $\mathbf{z}^{\pm}$ fields whose trace corresponds to the relative energy $E^R = \langle |\mathbf{v}|^2 - |\mathbf{b}|^2\rangle$, and is not an invariant of the MHD equations. Another von Kármán–Howarth equation can easily be written for $f^{+-}(r)$, but this is not done here, since this term is discarded in the inertial range on which we concentrate. The pressure term disappears from the above equation, since it involves a first-order solenoidal isotropic tensor equal to zero assuming regularity at $r=0$.

We now restate the above von Kármán–Howarth equation for MHD in terms of structure functions defined as usual as $B_{ij}^+(\mathbf{r}) \equiv \langle \delta z_i^+(\mathbf{r}) \delta z_j^+(\mathbf{r})\rangle$ and $B_{ijl}^{+-+}(\mathbf{r}) \equiv \langle \delta z_i^+(\mathbf{r}) \delta z_j^-(\mathbf{r}) \delta z_l^+(\mathbf{r})\rangle$, and defining as well the constant total flux of the + field as $\partial_t \langle z_L^{+^2}\rangle = -2\epsilon^+/d_*$; we further express the second- and third-order structure functions in terms of the correlation functions, namely,

$$R_{LL}^+(\mathbf{r}) = (1/d)\langle z_L^{+^2}\rangle - (\tfrac{1}{2}) B_{LL}^+(\mathbf{r}),$$

$$4\mathcal{C}_{LLL}^{+-+}(\mathbf{r}) = B_{LLL}^{+-+}(\mathbf{r}) - 2\mathcal{C}_{LLL}^{++-}(\mathbf{r}) = 4C_3 k^{+-+}, \quad (2)$$

where $C_{LLL}^{++-}(\mathbf{r}) \equiv \langle z_L^+(\mathbf{x}) z_L^+(\mathbf{x}) z_L^-(\mathbf{x}')\rangle$, and where homogeneity has been taken into account. We thus finally arrive at the alternative form of the von Kármán–Howarth equation for MHD where

$$-\frac{2\epsilon^+}{d} - \frac{1}{2}\frac{\partial B_{LL}^+(\mathbf{r})}{\partial t} = r^{-\gamma}\partial_r\left[r^{\gamma}\left(\frac{B_{LLL}^{+-+}(\mathbf{r})}{4} - \frac{\mathcal{C}_{LLL}^{++-}(\mathbf{r})}{2}\right)\right] - r^{-\gamma}\partial_r\{r^{\gamma}\partial_r[\nu_+ B_{LL}^+(\mathbf{r}) + \nu_- B_{LL}^{-+}(\mathbf{r})]\}, \quad (3)$$

with $\gamma = d+1$ and $B_{LL}^{-+}(\mathbf{r}) \equiv \langle \delta z_L^-(\mathbf{r}) \delta z_L^+(\mathbf{r})\rangle$. Neglecting the time-derivative term (as compared to ϵ^+) and the dissipative terms in the inertial domain, we now integrate over r and obtain the law analogous to that of Kolmogorov [5], but for magnetized flows, namely

$$\langle \delta z_L^{+^2}(\mathbf{r}) \delta z_L^-(\mathbf{r})\rangle - 2\langle z_L^+(\mathbf{x}) z_L^+(\mathbf{x}) z_L^-(\mathbf{x}')\rangle = -C_d \epsilon^+ r, \quad (4)$$

with $C_d = 2K_d/3$ and

$$d_*(d+2) K_d = 12; \quad (5)$$

$K_3 = \frac{4}{5}$ in the standard three-dimensional case where $d = d_* = 3$. Using the $\pm$ symmetry, the equivalent law for $\mathbf{z}^-$ immediately obtains

$$\langle \delta z_L^{-^2}(\mathbf{r}) \delta z_L^+(\mathbf{r})\rangle - 2\langle z_L^-(\mathbf{x}) z_L^-(\mathbf{x}) z_L^+(\mathbf{x}')\rangle = -C_d \epsilon^- r. \quad (6)$$

Note that for $\epsilon^+ \sim \epsilon^-$, i.e., for a negligible flux of velocity magnetic-field correlation ϵ^C (with $2\epsilon^C = \epsilon^+ - \epsilon^- = -2\dot{E}^C$, where $E^C = \langle \mathbf{v}\cdot\mathbf{b}\rangle$), the left-hand side of Eqs. (4) and (6) are comparable.

At this point, it is instructive to go back to the original physical variables of MHD, namely, the velocity and the magnetic field. One uses now, instead of $\epsilon^{\pm}$, the corresponding total energy and correlation fluxes ϵ^T and ϵ^C with $\epsilon^+ + \epsilon^- = 2\epsilon^T$. The two scaling laws in terms of the velocity and magnetic-field moments are [16]

$$\langle \delta v_L^3(\mathbf{r})\rangle - 6\langle b_L^2(\mathbf{x}) v_L(\mathbf{x}+\mathbf{r})\rangle = -K_d \epsilon^T r, \quad (7)$$

$$-\langle \delta b_L^3(\mathbf{r})\rangle + 6\langle v_L^2(\mathbf{x}) b_L(\mathbf{x}+\mathbf{r})\rangle = -K_d \epsilon^C r; \quad (8)$$

the Kolmogorov law [5] recovers for $\mathbf{b} \equiv 0$. Alternatively, in terms of correlation functions only, and noting that $\langle \delta b_L^3(\mathbf{r})\rangle = 6\langle b_L^2(\mathbf{x}) b_L(\mathbf{x}+\mathbf{r})\rangle$, these scaling laws can be written in term of the point-wise relative energy of the longitudinal components defined as $E_L^R(\mathbf{x}) = v_L^2(\mathbf{x}) - b_L^2(\mathbf{x})$, leading to $\langle v_L(\mathbf{x}+\mathbf{r}) E_L^R(\mathbf{x})\rangle = -(K_d/6)\epsilon^T r$, and similarly, $\langle b_L(\mathbf{x}+\mathbf{r}) E_L^R(\mathbf{x})\rangle = -(K_d/6)\epsilon^C r$.

In the general case with comparable velocity and magnetic fields, *only* cross correlators, as indicated in fact by a direct inspection of the nonlinearities of the MHD equations, scale as r; such conservation laws imply nonzero correlations between the $(\mathbf{v},\mathbf{b})$ fields, as for the scalar cases mentioned before [10,11].

The consequences of the exact scaling laws derived in this paper deserve a thorough study; numerical computations in both two and three dimensions are under way. It has been shown [17,18] that in a flow with $E^C \sim 0$, i.e., with weak $(\mathbf{v},\mathbf{b})$ correlations, the *spatial* correlation $\rho_C(\mathbf{x}) = 2\mathbf{v}(\mathbf{x})\cdot\mathbf{b}(\mathbf{x})/(|\mathbf{v}(\mathbf{x})|^2+|\mathbf{b}(\mathbf{x})|^2)$ is in fact quite strong locally; similarly, large scales and small scales have $(\mathbf{v},\mathbf{b})$ correlations of opposite polarities [18]. These are the correlations that are responsible for a local (as well as global) slowing down of the dynamical evolution of magnetized flows giving rise to a different scaling for MHD than for neutral fluids. Furthermore, because of the presence of two independent scaling functions for the two Elsässer variables, different scaling laws may arise for highly correlated flows, a result already known [18] at the level of second-order moments [19].

Preliminary results of direct numerical simulations in two space dimensions at low global correlation ($\rho_C \sim 0.05$) and with a slight excess of magnetic energy (by a factor ~ 3 in the statistically steady regime) indicate that in that case both longitudinal structure functions $\langle \delta z_L^{\pm^2} \delta z_L^{\mp} \rangle$ scale as r in the inertial range, and furthermore, that the third-order structure function of the velocity does not scale linearly with r, whereas that for the magnetic field does [20].

The extensions of the $\frac{4}{5}$ law of Kolmogorov for MHD derived here involve moments with different orders in the parameter $\chi_M = \langle|\mathbf{b}|^2\rangle/\langle|\mathbf{v}|^2\rangle$ and must be fulfilled for the whole range of values that this ratio can take. It is known [21] that MHD flows evolve asymptotically in time towards either of three possible regimes: the hydrodynamical regime where $\chi_M \sim 0$ is dominated by the velocity field, the opposite regime where χ_M is large, a regime corresponding to strongly magnetized plasmas as encountered in tokamaks or in the solar corona, and a third regime with $\chi_M \sim 1$. It is instructive to rewrite the scaling laws for MHD introducing normalizing factors which can be defined with the rms of any velocity and magnetic-field component, viz. $v^2 = \langle v_L^2 \rangle = \langle v_n^2 \rangle$ and $b^2 = \langle b_L^2 \rangle = \langle b_n^2 \rangle$. Denoting the adimensionalized third-order tensors with a ^ symbol, we now obtain

$$\widehat{\langle \delta v_L^3(\mathbf{r}) \rangle} - 6\chi_M \widehat{\langle b_L^2(\mathbf{x}) v_L(\mathbf{x}+\mathbf{r}) \rangle} = -K_d \frac{\epsilon^T}{v^3} r, \tag{9}$$

$$-\chi_M^{3/2} \widehat{\langle \delta b_L^3(\mathbf{r}) \rangle} + 6\chi_M^{1/2} \widehat{\langle v_L^2(\mathbf{x}) b_L(\mathbf{x}+\mathbf{r}) \rangle} = -K_d \frac{\epsilon^C}{v^3} r. \tag{10}$$

In each of these distinct regimes, some terms of the above relationships dominate and must scale as r, hence it can be conjectured that the combination of terms for each power of χ_M scales independently linearly with r.

Thus, the linear scaling in r holds for the structure function cubic in the magnetic field for magnetically dominated flows. It stems from the flux of $(\mathbf{v},\mathbf{b})$ correlation (with $\mathbf{v} \neq 0$) and leads in dimension three to

$$\langle \delta b_L^3(\mathbf{r}) \rangle = \tfrac{4}{5} \epsilon^C r. \tag{11}$$

In the solar wind, this relationship does not seem to apply [22], possibly an indication that the turbulence is not fully developed; however, using the left-hand side of Eq. (11) as the independent variable (instead of r itself) may yield better results for computing the anomalous scaling exponents of structure functions at all orders; this was shown experimentally for fluids at moderate Reynolds numbers [23]. This relationship implies that the skewness of the magnetic field (or more precisely the normalized third-order longitudinal structure function) is of the same sign as the flux of $(\mathbf{v},\mathbf{b})$ correlation. It is known empirically from both closure calculations and numerical computations that ϵ^C is of the same sign as E^C, i.e., that the correlation undergoes a direct transfer to small scales, albeit more slowly than the energy [24]. Hence, Eq. (11) implies that the skewness of $\mathbf{b}$ is of the same sign as the $(\mathbf{v},\mathbf{b})$ correlation in the strong magnetic-field regime.

On the other hand, in the context of the dynamo problem, assuming that initially the magnetic field is weak and thus neglecting its third-order correlator in Eq. (8), we deduce immediately that since correlations between the growing field and the velocity (that creates it) grow as $\mathbf{b}$ grows, this in turn affects the Kolmogorov scaling [5] of the velocity itself in Eq. (7); this indicates again the role of the velocity–magnetic-field correlation in the building up of an intermittent field, like the passive scalar that is known to be more intermittent than the velocity that carries it. Furthermore, when the correlation flux ϵ^C is weak compared to the flux of energy [a case that corresponds to a more efficient nonlinear transfer of energy towards small scales than of $(\mathbf{v},\mathbf{b})$ correlation, as observed in several models and numerical simulations], $\langle \delta b_L^3(\mathbf{r}) \rangle \sim \langle v_L^2(\mathbf{x}) b_L(\mathbf{x}+\mathbf{r}) \rangle$; note that this does not imply $E_L^R(\mathbf{x}) \sim 0$, since $\epsilon^T \neq 0$.

At this stage, a simplification can be made when $\chi_M \sim 1$ and thus when the normalized cross correlator between $\mathbf{z}^{\pm}$ fields defined as $2\langle z_i^+ z_i^- \rangle / \langle |\mathbf{z}^+|^2 + |\mathbf{z}^-|^2 \rangle = (1-\chi_M)/(1+\chi_M)$ is close to zero. Indeed, most of the time, the $\mathbf{z}^{\pm}$ fields evolve quasi-independently, traveling in opposite directions along the large-scale quasi-uniform magnetic field; one can then assume that they are decorrelated at different spatial locations, but of course are fully correlated when considered at the same point where they interact to give rise to the nonlinear coupling of MHD turbulence (this is the essence of the weakening of nonlinear steepening due to the presence of Alfvén waves [12]). Thus, the correlator $\langle z_i^+(\mathbf{x}) z_j^+(\mathbf{x}) z_l^-(\mathbf{x}') \rangle$ can be factorized with this hypothesis into $\langle z_i^+(\mathbf{x}) z_j^+(\mathbf{x}) \rangle \langle z_l^-(\mathbf{x}') \rangle = 0$. Note however that this does *not* imply that a correlator such as, for example, $\langle z_i^+(\mathbf{x}) z_j^+(\mathbf{x}) z_l^-(\mathbf{x}) z_m^-(\mathbf{x}') \rangle$ factorizes; of course this does not imply any factorization for $(\mathbf{v},\mathbf{b})$ moments either.

With that hypothesis, the above expressions simplify into a form more akin to the Kolmogorov law, since it involves only structure functions:

$$\langle \delta z_L^{+^2}(\mathbf{r}) \delta z_L^-(\mathbf{r}) \rangle = -C_d \epsilon^+ r, \tag{12}$$

$$\langle \delta z_L^{-^2}(\mathbf{r}) \delta z_L^+(\mathbf{r}) \rangle = -C_d \epsilon^- r. \tag{13}$$

In terms of the $(\mathbf{v},\mathbf{b})$ fields again, these restricted relationships now read $\langle \delta v_L^3(\mathbf{r}) \rangle - \langle \delta v_L(\mathbf{r}) \delta b_L^2(\mathbf{r}) \rangle = -C_d \epsilon^T r$, and similarly, $-\langle \delta b_L^3(\mathbf{r}) \rangle + \langle \delta b_L(\mathbf{r}) \delta v_L^2(\mathbf{r}) \rangle = -C_d \epsilon^C r$.

Finally, the calculations presented in this paper provide a set of surrogate lengths for MHD which are useful for data analysis, e.g., in terms of the correlators in Eqs. (4) and (6) because, as noted before, this allows for extended power-law ranges. Besides providing a guide to a phenomenological description of MHD flows, this should also be useful in analyzing, e.g., the solar wind data [25].

The extension of the present analysis to several variants of the MHD equations, including small-scale kinetic effects [26] such as the Hall current or the Pedersen resistivity—relevant, for example, to magnetospheric physics [27]—poses no particular problem and is left for future work; it will represent a more realistic step towards describing astrophysical or geophysical fluids.

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[15] Note that $d_* \neq d$, for example when the variation on a third dimension is assumed negligible and all d_* components of the vector fields are kept.

[16] Such laws do not apply in the case of exact solutions to the MHD equations like Alfvén waves $\mathbf{v} = \pm \mathbf{b}$, which render the nonlinear terms exactly equal to zero (except for the pressure), and cannot be considered within the present framework, since this analysis stems from the existence of a nonzero transfer of energy to small scales.

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